First Edition

Multivariable Calculus

Essentials

Content Guide with Questions

Ramesh

Preface

Welcome to Calculus 3, also known as multivariable calculus. Multivariable calculus will be an extension of the calculus you have been learning as you will perform operations, including differentiation and integration, on functions with more than one variable. Instead of performing vector calculus for values on a number line, you will be working with points in space.

This reference guide will allow you to understand Multivariable Calculus in a simple and concise format while it covers all the important formulas and concepts you must know for the course and the tests. To make the best use of this guide, practice with the section questions and check your answers with the provided answers.

The Unit Exams feature questions from actual administered exams and are great ways to prepare for the actual test.

Foreword

Avanthika Ramesh and Abhisri Ramesh were in my calculus class a year ago, and were some of the best students.
They have created this summary of Multivariate Calculus, and I highly recommend it.
Sit down with this summary and work though the problems.
If you can do that, you should do well in the course.

Sincerely,
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Vectors & Curves

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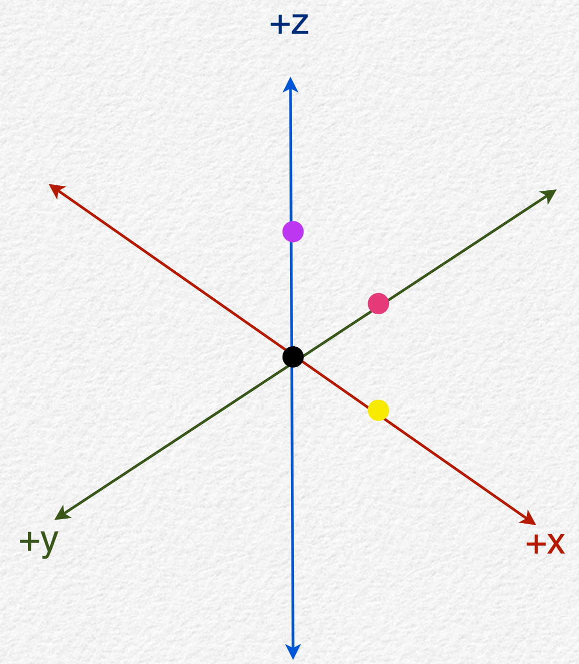
Vectors

Multivariable calculus revolves around the concept of vectors and vector calculations. Therefore, it is essential to grasp a thorough understanding of vectors.

Overview

3D Space

The 3D coordinate system contains the 2D xy axis, and an additional z axis that is perpendicular to both the x and y axis. The 3D coordinate system forms three planes - xy, yz, xz - which all intersect at the origin (0,0,0). All points in 3D space will have 3 values for their coordinates - the x, y and z coordinates respectively.



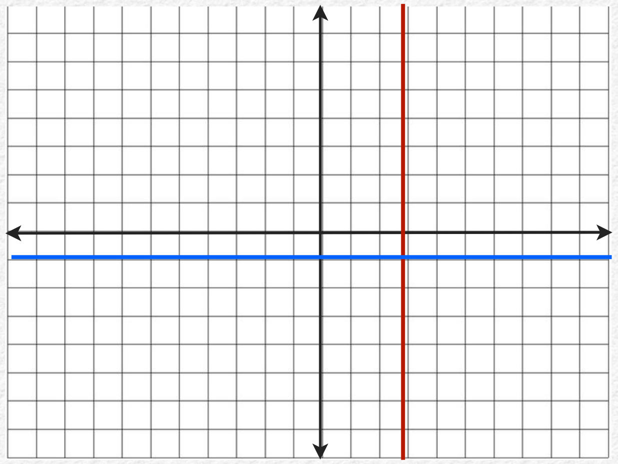
In the figure above, the positive x, y, and z axis are labeled. As shown, the z-axis is perpendicular to both the x and y axis.

The black dot in the center is the origin (0,0,0).

The coordinates for the point in purple, pink, and yellow are (0,0,5), (0,-5,0), and (5,0,0) respectively.

3D Equations

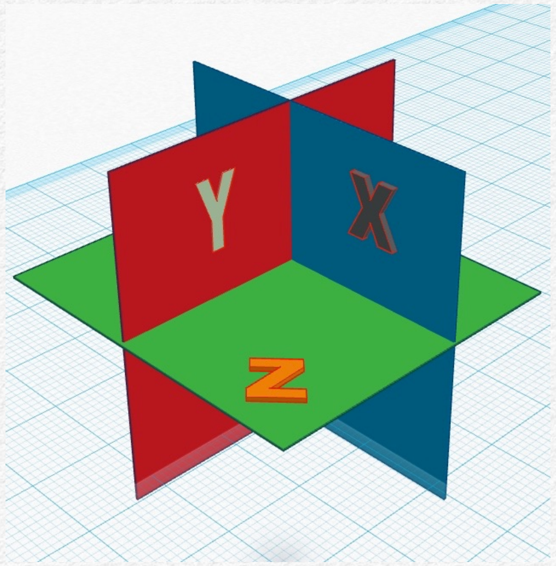
In 3D space, whenever x or y equaled a constant, the equation was represented by a line.



x = 3

y = -1

In 3D space, whenever x,y,or z is equal to a constant, the equation will be represented by a plane.



x = 0

y = 0

z = 0

Note that x^2 + y^2 = r^2 is a generic equation for a circle in 2D space. In 3D space, the geometric object is a sphere instead of a circle. A circle can still be generated by excluding one of the variables.

Below are some basic 3D formulas that are important to know:

Distance formula in 3D space:

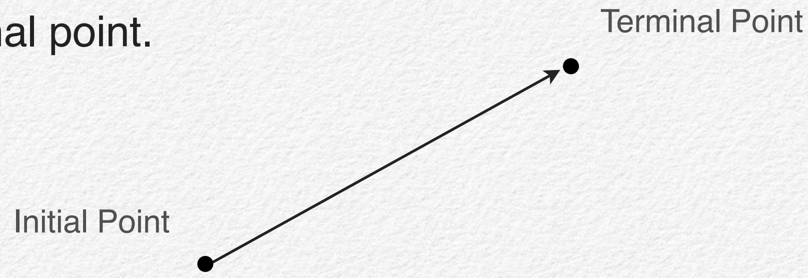
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Standard equation for a sphere with center (a,b,c) and radius r:

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Vectors

Vectors are rays that have a magnitude and a direction. The magnitude is the length of the vector. They have an initial point and a terminal point.



3 dimensional vectors have three components <i, j, k>. Take the vector <x,y,z>. The x value is the i component. The y value is the j component. The z value is the k component.

Vectors can be represented in different notations. The vector <x,y,z> can also be written as xi + yj + zk. For example, <43,52,32> can be written as 43i + 52j + 32k.

The component form of a 3D vector is computed the same way as for a 2D vector with an additional z coordinate.

Consider a vector with terminal point (x1, y1, z1) and initial point (x0,y0,z0).
The component form of the vector is <x1-x0, y1-y0, z1-z0>.

Vector Equations

Below are important equations to know to perform vector operations.
Magnitude (computes the length of a vector)
The magnitude of vector <u1,u2,u3> is

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Direction (unit vector formula)
*Note that each component of vector a (i, j, & k components) must be divided by the magnitude, IaI.

$$unit\ vector = \frac{\underline{a}}{\underline{|a|}}$$

Midpoint

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Dot Product
The dot product is an operation performed on vectors that returns a scalar (a number value).
Given vectors a and b, the dot product is:

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\vec{a}\| \cdot \left\|\vec{b}\right\| \cos\theta$$

Cross Product

The cross product is an operation performed on vectors that returns a vector. The result of a cross product is a vector that is perpendicular to both vectors. Given vectors A and B, the cross product is computed by

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\vec{i} - (A_x B_z - A_z B_x)\vec{j} + (A_x B_y - A_y B_x)\vec{k}$$

First, get both vectors into matrix format by designating the i, j, and k components to the 3 columns of the first row. Then designate the i,j,and k components of the first vector to the second row, and the second vector to the third row, as shown above.

Then, cover up the i column and take the determinant of the 2x2 matrix under the j and k column. This is the i component of the resulting vector. Similarly, cover up the j column and compute the determinant of the 2x2 matrix shown, which includes the 2 values under the i column and the 2 values under the k column. Repeat this procedure for the k column by covering it up and compute the final cross product vector. Note the signs of each component (the j component is subtracted).

Vector Projection

When dealing with vectors, it is essential to know the magnitude of a vector in a certain direction. The projection of a vector onto another vector will tell you the magnitude of the original vector in a different vector's direction.

Here, the vector u is projected onto vector v.

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

It is important to know which vector is being projected onto another vector in order to perform the correct operations on each vector.

Angle Between Vectors

Sometimes, the angle theta may not be known. Using algebraic techniques, the formula for the projection of a vector can be manipulated in terms of theta, which is the angle between the vectors.

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$$

Notice how the dot product is computed in the numerator, and then divided by the product of the magnitudes of both vectors.

Scalar Projection of a Vector

The scalar projection determines the length of a vector projected onto another vector. Given that vector b is projected on vector a, the scalar projection is:

$$\frac{\vec{a}.\vec{b}}{|\vec{a}|}$$

Vector Geometry

Vector operations can be used to determine if vectors are parallel or perpendicular, and find the area and volume of figures formed by multiple vectors.

Before introducing vector geometry, it is important to emphasize what a nonzero vector is.

A **nonzero** vector is any vector whose components are not all zeros. For example, <0,0> and <0,0,0> are both zero vectors since every component is zero. However, <4,0,0> is a nonzero vector as not all of its components are zero.

Orthogonal Vectors

Two vectors are perpendicular, or orthogonal, if the angle between the nonzero vectors is pi/2. **For orthogonal vectors, the dot product = 0.** Hence to test if two vectors are perpendicular/orthogonal, compute the dot product and see if its equals 0.

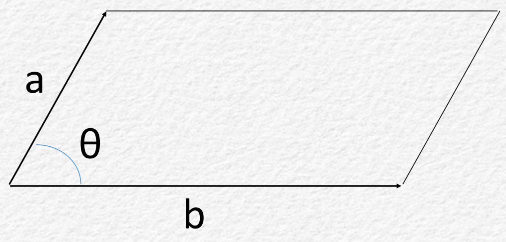
Parallel Vectors

The cross product determines if two vectors are parallel. The cross product of two vectors is zero if both vectors are parallel to each other, OR if one of the vectors is a zero vector.

Additionally, if the angle between two vectors = 0, the vectors are parallel. If the angle = pi, they vectors are anti-parallel.

Area of a Parallelogram

A parallelogram is formed by the connection of 4 vectors, of which 2 vectors are distinct. The area of the parallelogram can be computed by finding the magnitude of the cross product of the two vectors that form the parallelogram. Here, a parallelogram is formed by vectors a and b, and the area is determined by: $|a \times b|$.



Area of a Triangle

From geometry, we know that is area of a triangle is half the area of a rectangle (which is also a parallelogram). Therefore, the formula for the area of a triangle ABC is

$$A = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$$

The Triple Scalar Product

The formula for the triple scalar product of three vectors is:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \end{aligned}$$

The triple scalar product is the volume of a parallelepiped, which is a 3D figure whose 6 faces are parallelograms.

Section Questions

1. If vector $\overrightarrow{AB}$ = i + 4j – 2k, and point B is (5, 1, 3), find point A.

2. Find the angles between the vectors:
   u = 2i - 2j + k; v = 3i + 4k

3. Find the area of the parallelogram with the following vertices:
   A(0,0,0); B(3,2,4); C(5,1,4); D(2,-1,0)

Section Answers

1. A = (4,-3,5)

2. .75 radians

3. $\sqrt{129}$

Lines In Space

Overview

1. **Parametrization of a Line**

Parametrization of a Line

The parametrization of a line is determined by a point (to give location) and a vector (to give direction).

Given a point (a,b,c) and vector <d,e,f>, the parametric equations are:

x= a + di

y= b + ej

z = c + fk

*Recall that <d,e,f> = di + ej + fk

Parametrizing the line segment joining two points P and Q:

1) Compute vector PQ by subtracting the x , y , and z coordinates of the points

2) If P = (A,B,C) and Q = (D,E,F), vector PQ = (D-A, E-B, F-C)

3) Find the x,y, and z parametric equations using the coordinates of point P and components of vector PQ, as highlighted above.

Section Questions

1. Find the parametric equation for the line through point (3,-4,-1) parallel to vector i + j + k.

2. Find the parametrization for the line segments joining the point (0,1,1) and (0,-1,1).

3. Find the equation of the plane through (0,2,-1) normal to $n = 3i - 2j - k$.

Section Answers

1. $x = 3 + t$; $y = -4 + t$; $z = -1 + t$

2. $x = 0$; $y = 1 - 2t$; $z = 1$

3. $3x - 2y - z = -3$

Quadric Surfaces

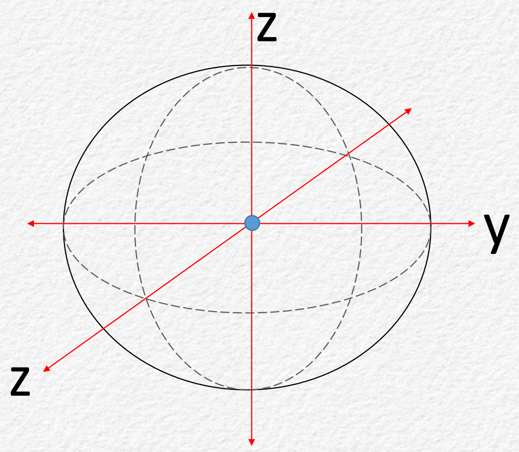
Overview

1. Types of Quadric Surfaces

Types of Quadric Surfaces

Quadric surfaces include more types of surfaces apart from conics and planes. The most important quadric surfaces to know are shown below. It is important to know the shape and the general formula of each quadric surface.

Ellipsoid

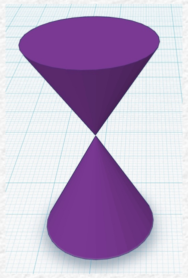


General Formula:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

For an ellipsoid, if a = b = c, then the quadric surface forms a sphere.

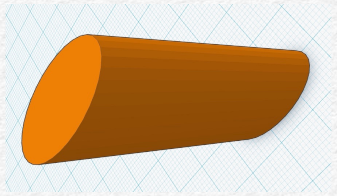
Cone



General Formula:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

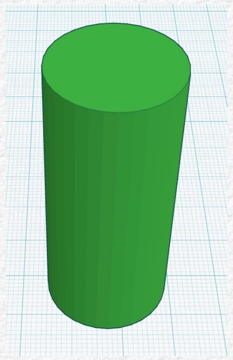
Elliptic Cylinder



General Formula:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

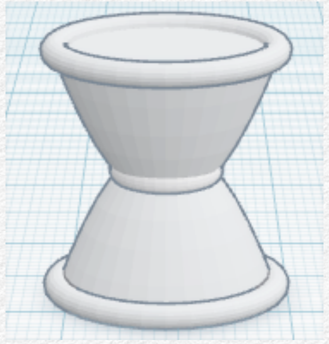
Circular Cylinder



General Formula:

$$x^2+y^2=r^2$$

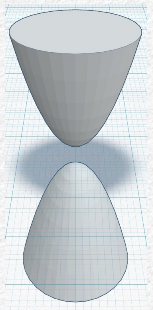
Hyperboloid of One Sheet:



General Formula:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

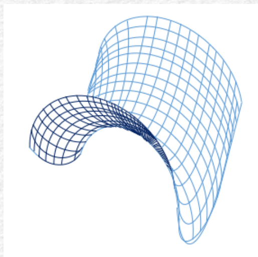
Hyperboloid of Two Sheets:



General Formula:

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

Hyperbolic Paraboloid



General Formula:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$$

Section Questions

1. Identify the quadric surface:
   $x^2 + 7y^2 + 4z = 20$

2. Identify the quadric surface:
   $x = z^2 - 7y^2$

3. Identify the quadric surface:

$x^2 + 3z^2 = 5y^2$

Section Answers

1. Ellipsoid
2. Hyperbolic Paraboloid
3. Cone

Curves

Overview

1. **Position Vectors**
2. **Tangential Vectors**
3. **Acceleration Vectors**

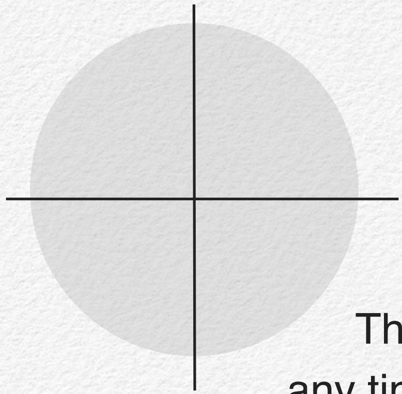
Position Vectors

A certain particle's path forms a curve in 3D space. The particle's path can be defined by multiple equations over a time interval t. The equations that define the particle's path and the time interval of its travel parametrize the curve of that particle's path.

The position vector for a particle moving in 3D space over time is defined by r(t) = ...

Where the exact position of the particle at a certain time is P(f(t),g(t),h(t))

Component Functions*** are the f(t) g(t) h(t) functions of the position vector



The circle is a common function represented as the position vector r(t)= cos(t)i + sin(t)j

Thus, the exact position of a particle at any time t is found by plugging in that value of t into the position vector function

Tangential Vectors

The tangent vector to the curve is the same as a particle's velocity vector V(t).

V(t) = dr/dt, and is simply calculated by taking the derivative of each component of the position vector function.

The direction of the velocity vector at any time t during the particles motion indicates the direction of the particle's motion (v/lvl)
*** notice that this direction is a unit vector

Speed: The magnitude of v, which is the velocity vector, calculates the particle's speed. Speed is represented by ds/dt, which equal to lvl.

Acceleration Vectors

The acceleration vector can be computed by taking the derivative of each component of the velocity vector OR by taking the second derivative of each component of the position vector.

Antiderivative Vectors

The antiderivative R(t) of a vector r(t) is: R(t) = r(t) + C
(take the integral of each vector component)

This comes useful when solving for position given a velocity vector or velocity given an acceleration vector. The initial condition provided will help solve for the antiderivative vector.

Arc length

The length of a curve from one point to another is defined as the following formula:
(x' = dx/dt; y'=dy/dt…)
Length: Int(sqrt(x'^2+y'^2+z'^2)dt) taken from t=a to t=b
This formula is equivalent to taking the integral of lvl from t=a to t=b.

Unit Tangent Vector (T)

T, or the unit tangent vector, follows this formula:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

T follows in the same direction as v (velocity), but differs in that it is always a unit vector.

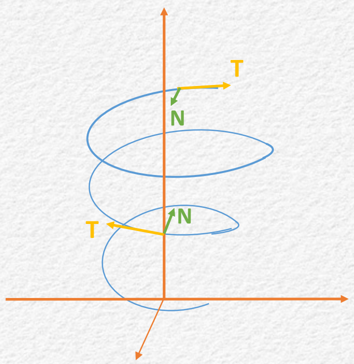
Principal Unit Normal (N)

The Principle Unit Normal, N, points in the direction of the path that the curve will follow.
The general formula is:

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}$$

This vector, N, is perpendicular to the curve.

The vectors T and N are represented below:



Curvature (K)

The formula for curvature K of a smooth curve is:

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

The formulas below are alternatives for determining curvature:

$$\kappa = \frac{\left\|\vec{T}'(t)\right\|}{\left\|\vec{r}'(t)\right\|} \qquad \kappa = \frac{\left\|\vec{r}'(t) \times \vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|^3}$$

Radius of Curvature:The radius of curvature is equal to 1/(curvature).

A large curvature value implies that the curve is turning sharply and has a sharp bend, while a lower curvature values implies more smoother curves with less of a bend.

Binormal Vector

The binormal vector, B, is calculated by taking the cross product of the Unit Tangent Vector and the Principal Unit Normal Vector.

The formula is:
B = T x N

Torsion

Torsion represents the “twisting” of the curve and is known as dB/ds which simplifies to –(dB/ds)* N.
The most common formula for torsion can seem quite complicated, but may come in handy.

First compute the triple scalar product of:
Ix’ y’ z’ I
Ix” y” z”I
Ix”’ y”’ z”’ I
Then, divide that answer by I v X a I^2

(x’= dx/dt, x”= d2x/dt^2, x”’= d3x/dt^3, y’= dy/dt…)

Another Acceleration Formula:

Acceleration can be computed using another formula:

$a = a_N N + a_T T$

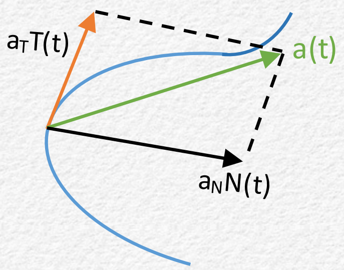
The tangential component of acceleration, a_T is:

$$a_T = a \bullet T = \frac{v \bullet a}{\|v\|}$$

The normal component of acceleration, a_N is:

$$a_N = a \bullet N = \frac{\|v \times a\|}{\|v\|}$$

The coefficient for T tells us how much the length of the velocity vector is changing, while the coefficient for N tells us how much the direction of the velocity vector is changing.



Another way to find acceleration, a, is:

$a^2 = a_N{}^2 + a_T T^2$

Summary

Position (r(t))	<x,y,z>
Velocity	<x',y',z'>
Acceleration	<x",y",z">
Arc length	∫ (|v(t)|) from [a,b]
Unit Tangent	v/ |v|
Curvature	(1/|v|)*(|dT/dt|) or |dT/ds|
Principal Normal Vector	T'/ |T'|
Binormal Vector	T X N
Acceleration	$a = a_N N + a_T T$
Torsion	|x' y' z' | |x" y" z" | |x"' y"' z"' | ________________ | v X a |^2 (x'= dx/dt, x"= d2x/dt^2, x"'= d3x/dt^3, y'= dy/dt…)

Section Questions

1. Find the velocity vector, acceleration vector, speed, and direction: $r(t) = (t+1)i + (t^2-1)j + 2tk$; $t = 1$

2. Find the length of the curve $r(T) = 4\cos(t)i + 4\sin(t)j + 3tk$ from $t=0$ to $t= \pi/2$

3. Find T, N, and k (curvature) for $r(t) = 3\text{sint}(t)i + 3\cos(t)j + 4tk$.

Section Answers

1. $v(t) = i + 2tj + 2k$; $a(t) = 2j$; speed = 3; direction: $1/3i + 2/3j + 2/3k$

2. $5\pi/2$

3. $T = 3/5 \cos(t)i – 3/5 \sin(t)j + 4/5k$; $N = -\sin(t)i – \cos(t)j$; $k = 3/25$

Unit Exam

1. Let $u=2i-3j+4k$

(a) Find the area of the parallelogram spanned by u and v

(b) Find the volume of the parallelepiped with sides touching u,v, and w

(c) Two projectiles follow the paths:

$r(t)= (t^2 +2t)i+tj+(100-t^2)k$

$r2(t)=(2t^3-3t)i +tj+ 3t^2 k$

When will the projectiles collide, if they do?

2.

(a) Consider the curve parametrized by:

$r(t)= ti+5j+(\frac{1}{2} t^2)k$

at t=2, find the Unit tangent T, unit normal N, and curvature k

3. Find the tangential and normal components of acceleration for the motion

$r(t)=t^2i +tj+tk$